The general Solow model

• Back to a closed economy.
• In the basic Solow model: no growth in GDP per worker in steady state. This contradicts the empirics for the Western world (stylized fact #5). In the general Solow model:
  • Total factor productivity, $B_t$, is assumed to grow at a constant, exogenous rate (the only change). This implies a steady state with balanced growth and a constant, positive growth rate of GDP per worker.
  • The source of long run growth in GDP per worker in this model is **exogenous** technological growth. Not deep, but:
    – it’s not trivial that the result is **balanced** growth in steady state,
    – reassuring for applications that the model is in accordance with a fundamental empirical regularity.
• Our focus is still:
  what creates economic progress and prosperity...
The micro world of the Solow model

... is the same as in the basic Solow model, e.g.:

- The same **object** (a closed economy).
- The same **goods** and **markets**. Once again, markets are competitive with real prices of $1, r_t$ and $w_t$, respectively. There is one type of output (one sector model).
- The same **agents**: consumers and firms (and government), essentially with the same behaviour, specifically: one representative profit maximising firm decides $K_t^d$ and $L_t^d$ given $r_t$ and $w_t$.
- One difference: **the production function**. There is a possibility of technological progress:

$$Y_t = B_t K_t^\alpha L_t^{1-\alpha}, \quad 0 < \alpha < 1.$$  

The full sequence $(B_t)$ is exogenous and $B_t > 0$ for all $t$. Special case is $B_t = B$ (basic Solow model).
The production function with technological progress

\[ Y_t = B_t K_t^{\alpha} L_t^{1-\alpha} \quad \text{with a given sequence, } (B_t) \leftrightarrow \]
\[ Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} \quad \text{with a given sequence, } (A_t), \quad A_t \equiv B_t^{1/(1-\alpha)}. \]

• With a Cobb-Douglas production function it makes no difference whether we describe technical progress by a sequence, \((B_t)\), for TFP or by the corresponding sequence, \((A_t)\), for labour augmenting productivity.

• In our case the latter is the most convenient. The exogenous sequence, \((A_t)\), is given by:

\[ A_{t+1} = (1 + g) A_t, \quad g > -1 \]
\[ \Rightarrow A_t = (1 + g)^t A_0, \quad g > -1 \]

• Technical progress comes as “manna from heaven” (it requires no input of production).
• Remember the definitions: \( y_t \equiv Y_t / L_t \) and \( k_t \equiv K_t / L_t \).

• Dividing by \( L_t \) on both sides of \( Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \) gives the per capita production function:

\[
y_t = k_t^\alpha A_t^{1-\alpha}.
\]

• From this follows:

\[
\ln y_t - \ln y_{t-1} = \alpha (\ln k_t - \ln k_{t-1}) + (1-\alpha)(\ln A_t - \ln A_{t-1}) \Leftrightarrow \\
g_t^y = \alpha g_t^k + (1-\alpha) g_t^A \equiv \alpha g_t^k + (1-\alpha) g.
\]

Growth in \( y_t \) can come from exactly two sources, and \( g_t^y \) is the weighted average of \( g_t^k \) and \( g \) with weights \( \alpha \) and \( 1-\alpha \).

• If, as in balanced growth, \( k_t / y_t \) is constant, then \( g_t^y = g \)!
The complete Solow model

\[ Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \]

\[ r_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} = \alpha \left( \frac{K_t}{A_t L_t} \right)^{\alpha-1} \]

\[ w_t = (1-\alpha) K_t^\alpha L_t^{-\alpha} A_t^{1-\alpha} = (1-\alpha) \left( \frac{K_t}{A_t L_t} \right)^\alpha A_t \]

\[ S_t = s Y_t \]

\[ K_{t+1} - K_t = S_t - \delta K_t , \quad K_0 \text{ given} \]

\[ L_{t+1} = (1+n) L_t , \quad L_0 \text{ given} \]

\[ A_{t+1} = (1+g) A_t , \quad A_0 \text{ given} \]

- **Parameters:** \( \alpha, s, \delta, n, g \). Let \( g > 0 \).
- **State variables:** \( K_t, L_t \) and \( A_t \).
- **Full model?** Yes, given \( K_0, L_0 \) and \( A_0 \) the model determines the full sequences \( (K_t), (L_t), (A_t), (Y_t), (r_t), (w_t), (S_t) \).
• **Note:**

\[ r_t K_t = \alpha K_t^\alpha (A_t L_t)^{1-\alpha} = \alpha Y_t \]
\[ w_t L_t = (1-\alpha) K_t^\alpha (A_t L_t)^{1-\alpha} = (1-\alpha) Y_t. \]

That is: capital’s share \( = \alpha \), labour’s share \( = 1-\alpha \), pure profits \( = 0 \). Our \( \alpha \) should still be around \( 1/3 \).

• **Also note:** defining “effective labour input” as \( \tilde{L}_t = A_t L_t \):

\[ \tilde{L}_{t+1} = (1+n)(1+g) \tilde{L}_t \equiv (1+\tilde{n}) \tilde{L}_t. \]

The model is mathematically equivalent to the basic Solow model with \( \tilde{L}_t \) taking the place of \( L_t \), and \( \tilde{n} \) taking the place of \( n \), and with \( B = 1 \)!

We could, in principle, take over the full analysis from the basic Solow model, but we will nevertheless be...
Analyzing the general Solow model

• If the model implies convergence to a steady state with balanced growth, then in steady state \( k_t \) and \( y_t \) must grow at the same constant rate (recall again that \( k_t / y_t \) is constant under balanced growth). Remember also:

\[
g_t^y = \alpha g_t^k + (1 - \alpha) g_t^A.
\]

Hence if \( g_t^y = g_t^k \), then \( g_t^y = g_t^k = g_t^A \). If there is convergence towards a steady state with balanced growth, then in this steady state \( k_t \) and \( y_t \) will both grow at the same rate as \( A_t \), and hence \( k_t / A_t \) and \( y_t / A_t \) will be constant.

• Furthermore: from the above mentioned equivalence to the basic Solow model, \( K_t / \tilde{L}_t = K_t / (A_t L_t) = k_t / A_t \) and \( Y_t / \tilde{L}_t = Y_t / (A_t L_t) = y_t / A_t \) converge towards constant steady state values.

• Each of the above observations suggests analyzing the model in terms of:
1. \( \tilde{k}_t \equiv \frac{k_t}{A_t} = \frac{K_t}{A_t L_t} \) and \( \tilde{y}_t \equiv \frac{y_t}{A_t} = \frac{Y_t}{A_t L_t} \).

2. From \( Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \) we get \( \tilde{y}_t = \tilde{k}_t^\alpha \).

3. From \( K_{t+1} - K_t = S_t - \delta K_t \) and \( S_t = sY_t \) we get
   \[
   K_{t+1} = sY_t + (1-\delta) K_t
   \]

4. Dividing by \( A_{t+1} L_{t+1} = (1+g)(1+n) A_t L_t \) on both sides gives
   \[
   \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)} \left( s\tilde{y}_t + (1-\delta) \tilde{k}_t \right).
   \]

5. Inserting \( \tilde{y}_t = \tilde{k}_t^\alpha \) gives the transition equation:
   \[
   \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)} \left( s\tilde{k}_t^\alpha + (1-\delta) \tilde{k}_t \right).
   \]

6. Subtracting \( \tilde{k}_t \) from both sides gives the Solow equation:
   \[
   \tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)} \left( s\tilde{k}_t^\alpha - (n+g+\delta+ng) \tilde{k}_t \right).
   \]
Convergence to steady state: the transition diagram

- The transition equation is:
  \[
  \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)} \left( s\tilde{k}_t^\alpha + (1-\delta)\tilde{k}_t \right).
  \]

- It is everywhere increasing and passes through (0,0).

- The slope of the transition curve at any \( \tilde{k}_t \) is:
  \[
  \frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \frac{s\alpha\tilde{k}_t^{\alpha-1} + (1-\delta)}{(1+n)(1+g)}.
  \]

- We observe: \( \lim_{\tilde{k}_t \to 0} \frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \infty \). Furthermore, \( \lim_{\tilde{k}_t \to \infty} \frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} < 1 \iff n + g + \delta + ng > 0 \). We assume that the latter very plausible stability condition is fulfilled.
• The transition equation must then look as follows:
• Convergence of $\tilde{k}_t$ to the intersection point $\tilde{k}^*$ follows from the diagram. Correspondingly: $\tilde{y}_t \to \tilde{y}^* = (\tilde{k}^*)^\alpha$.

Some first conclusions are:

• In the long run, $\tilde{k}_t \equiv k_t / A_t$ and $\tilde{y}_t = y_t / A_t$ converge to constant levels, $\tilde{k}^*$ and $\tilde{y}^*$, respectively. These levels define steady state.

• In steady state, $k_t$ and $y_t$ both grow at the same rate as $A_t$, that is, at the rate $g$ and the capital output ratio, $K_t / Y_t = k_t / y_t$, must be constant.
Steady state

• The Solow equation

\[ \tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)} \left( s\tilde{k}_t^\alpha - (n + g + \delta + ng) \tilde{k}_t \right) \]

together with \( \tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^* \) gives:

\[ \tilde{k}^* = \left( \frac{s}{n + g + \delta + ng} \right)^{1-\alpha} \Rightarrow \tilde{y}^* = \left( \frac{s}{n + g + \delta + ng} \right)^{1-\alpha} . \]

• Using \( \tilde{k}_t \equiv k_t / A_t \) and \( \tilde{y}_t \equiv y_t / A_t \) we get the steady state growth paths:

\[ k_t^* = A_t \left( \frac{s}{n + g + \delta + ng} \right)^{1-\alpha} \quad \text{and} \quad y_t^* = A_t \left( \frac{s}{n + g + \delta + ng} \right)^{1-\alpha} . \]
• Since \( c_t = (1-s) y_t \),
\[
c_t^* = A_t (1-s) \left( \frac{s}{n + g + \delta + ng} \right)^{\frac{\alpha}{1-\alpha}}.
\]

• It also easily follows from
\[
r_t = \alpha (\tilde{k}_t)^{\alpha-1} \quad \text{and} \quad w_t = (1-\alpha) A_t (\tilde{k}_t)^{\alpha}
\]
that
\[
r^* = \alpha \left( \frac{s}{n + g + \delta + ng} \right)^{-1} \quad \text{and} \quad w^*_t = A_t (1-\alpha) \left( \frac{s}{n + g + \delta + ng} \right)^{\frac{\alpha}{1-\alpha}}.
\]

• **There is balanced growth in steady state:** \( k_t, y_t \) and \( w_t \) grow at the same constant rate, \( g \), and \( r_t \) is constant.

• **There is positive growth in GDP per capita in steady state** (provided that \( g > 0 \)).
Structural policies for steady state

- Output per capita and consumption per capita in steady state are:
  \[ y_t^* = A_0 (1 + g)^t \left( \frac{s}{n + g + \delta + ng} \right)^{\frac{\alpha}{1-\alpha}} \]
  \[ c_t^* = A_0 (1 + g)^t (1 - s) \left( \frac{s}{n + g + \delta + ng} \right)^{\frac{\alpha}{1-\alpha}} \].

- Golden rule: the \( s \), that maximises the entire path, \( c_t^* \). Again: \( s^{**} = \alpha \).

- The elasticities of \( y_t^* \) wrt. \( s \) and \( n + g + \delta \) are again \( \alpha / (1 - a) \) and \( -\alpha / (1 - a) \), respectively.

- Policy implications from steady state are as in the basic Solow model: encourage savings and control population growth.

- **But** we have a new parameter, \( g \) (\( A_0 \) corresponds to \( B \)). We want a large \( g \), but it is not easy to derive policy conclusions wrt. technology enhancement from our model (\( g \) is exogenous).
Empirics for steady state

\[ y_t^* = A_t \left( \frac{s}{n + g + \delta + ng} \right)^{\frac{\alpha}{1-\alpha}} \Rightarrow \]

\[ \ln y_t^* = \ln A_t + \frac{\alpha}{1-\alpha} \left[ \ln s - \ln (n + g + \delta + ng) \right]. \]

- Assume that all countries are in steady state in 2000!
- It’s hard to get good data for \( A_t \), so make the heroic assumption that \( A_t \) is the same for all countries in 2000.
- Set (plausibly) \( g + \delta \equiv 0.075 \).
- If \( y_{00}^i \) is GDP per worker in 2000 of country \( i \), the above equation suggests the following regression equation:

\[ \ln y_{00}^i = \gamma_0 + \gamma_1 \left[ \ln s^i - \ln \left( n^i 0.075 \right) \right], \]

with \( s^i \) and \( n^i \) measured appropriately (here as averages over 1960-2000), and where \( \gamma_1 = \alpha / (1 - \alpha) \).
An OLS estimation across 86 countries gives:

\[
\ln y_{00}^i = 8.812 + 1.47 \left( \ln s^i - \ln \left( n^i + 0.075 \right) \right), \quad \text{adj. } R^2 = 0.55
\]
• High significance! Large $R^2$! Even though we have assumed that $A_{00}$ is the same in all countries!
• But always remember the problem of correlation vs. causality.
• Furthermore: the estimated value of $\gamma$ is not in accordance with the theoretical (model-predicted) value of $1/2$. Or:

$$\frac{\alpha}{1 - \alpha} = 1.47 \Leftrightarrow \alpha = 0.60.$$ 

• The conclusion is mixed: the figure on the previous slide is impressive, but the figure’s line is much steeper than the model suggests.