Chapter 8

A Cash-In-Advance Model

Many macroeconomic approaches to modeling monetary economies proceed at a higher level than in monetary models with search or overlapping generations (one might disparagingly refer this higher level of monetary model as implicit theorizing). One approach is to simply assume that money directly enters preferences (money-in-the-utility-function models) or the technology (“transactions cost” models). Another approach, which we will study in this chapter, is to simply assume that money accumulated in the previous period is necessary to finance current period transactions. This cash-in-advance approach was pioneered by Lucas (1980, 1982), and has been widely-used, particularly in quantitative work (e.g. Cooley and Hansen 1989).

8.1 A Simple Cash-in-Advance Model With Production

In its basic structure, this is a static representative agent model, with an added cash-in-advance constraint which can potentially generate dynamics. The representative consumer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(n_s^t)] ,$$

where $0 < \beta < 1$, $c_t$ is consumption, and $n_s^t$ is labor supply. Assume that $u(\cdot)$ is strictly increasing, strictly concave, and twice differentiable,
and that \( v(\cdot) \) is increasing, strictly convex, and twice differentiable, with \( v'(0) = 0 \) and \( v'(h) = \infty \), where \( h \) is the endowment of time the consumer receives each period.

The representative firm has a constant-returns-to-scale technology,

\[
y_t = \gamma_t n_t^d,
\]

where \( y_t \) is output, \( n_t^d \) is labor input, and \( \gamma_t \) is a random technology shock.

Money enters the economy through lump-sum transfers made to the representative agent by the government. The government budget constraint takes the form

\[
\bar{M}_{t+1} = \bar{M}_t + P_t \tau_t,
\]

where \( \bar{M}_t \) is the money supply in period \( t \), \( P_t \) is the price level (the price of the consumption good in terms of money) and \( \tau_t \) is the lump-sum transfer that the representative agent receives in terms of consumption goods. Assume that

\[
\bar{M}_{t+1} = \theta_t \bar{M}_t,
\]

where \( \theta_t \) is a random variable.

In cash-in-advance models, the timing of transactions can be critical to the results. Here, the timing of events within a period is as follows:

1. The consumer enters the period with \( M_t \) units of currency, \( B_t \) one-period nominal bonds, and \( z_t \) one-period real bonds. Each nominal bond issued in period \( t \) is a promise to pay one unit of currency when the asset market opens in period \( t + 1 \). Similarly, a real bond issued in period \( t \) is a promise to pay one unit of the consumption good when the asset market opens in period \( t + 1 \).

2. The consumer learns \( \theta_t \) and \( \gamma_t \), the current period shocks, and receives a cash transfer from the government.

3. The asset market opens, on which the consumer can exchange money, nominal bonds, and real bonds.

4. The asset market closes and the consumer supplies labor to the firm.
5. The goods market opens, where consumers purchase consumption goods with cash.

6. The goods market closes and consumers receive their labor earnings from the firm in cash.

The consumer’s problem is to maximize (8.1) subject to two constraints. The first is a “cash-in-advance constraint,” i.e. the constraint that the consumer must finance consumption purchases and purchases of bonds from the asset stocks that she starts the period with,

\[ S_t B_{t+1} + P_t q_t z_{t+1} + P_t c_t \leq M_t + B_t + P_t z_t + P_t \tau_t. \]  

(8.5)

Here, \( S_t \) is the price in units of currency of newly-issued nominal bond, and \( q_t \) is the price in units of the consumption good of a newly-issued real bond. The second constraint is the consumer’s budget constraint,

\[ S_t B_{t+1} + P_t q_t z_{t+1} + M_{t+1} + P_t c_t \leq M_t + B_t + P_t z_t + P_t \tau_t + P_t w_t n^*_t, \]  

(8.6)

where \( w_t \) is the real wage. To make the consumer’s dynamic optimization stationary, it is useful to divide through constraints (8.5) and (8.6) by \( \bar{M}_t \), the nominal money supply, and to change variables, defining lower-case variables (except for previously-defined real variables) to be nominal variables scaled by the nominal money supply, for example \( p_t \equiv \frac{P_t}{\bar{M}_t} \). Constraints (8.5) and (8.6) can then be rewritten as

\[ S_t b_{t+1} \theta_t + p_t q_t z_{t+1} + p_t c_t \leq m_t + b_t + p_t z_t + p_t \tau_t \]  

(8.7)

and

\[ S_t b_{t+1} \theta_t + p_t q_t z_{t+1} + m_{t+1} \theta_t + p_t c_t \leq m_t + b_t + p_t z_t + p_t \tau_t + p_t w_t n^*_t \]  

(8.8)

Note that we have used (8.4) to simplify (8.7) and (8.8). The constraint (8.8) will be binding at the optimum, but (8.7) may not bind. However, we will assume throughout that (8.7) binds, and later establish conditions that will guarantee this.

The consumer’s optimization problem can be formulated as a dynamic programming problem with the value function \( v(m_t, b_t, z_t, \theta_t, \gamma_t) \). The Bellman equation is then

\[
\max_{c_t, n^*_t, m_{t+1}, b_{t+1}, z_{t+1}} \left[ u(c_t) - v(n^*_t) + \beta E_t v(m_{t+1}, b_{t+1}, z_{t+1}, \theta_{t+1}, \gamma_{t+1}) \right]
\]
subject to (8.7) and (8.8). The Lagrangian for the optimization problem on the right-hand side of the Bellman equation is

\[
\mathcal{L} = u(c_t) - v(n^s_t) + \beta E_t v(m_{t+1}, b_{t+1}, z_{t+1}, \theta_{t+1}) + \lambda_t (m_t + b_t + p_t z_t + p_t \tau_t - S_t b_{t+1} \theta_t - p_t q_t z_{t+1} - p_t c_t) + \mu_t (m_t + b_t + p_t z_t + p_t \tau_t + p_t w_t n^s_t - S_t b_{t+1} \theta_t - p_t q_t z_{t+1} - m_{t+1} \theta_t - p_t c_t),
\]

where \(\lambda_t\) and \(\mu_t\) are Lagrange multipliers. Assuming that the value function is differentiable and concave, the unique solution to this optimization problem is characterized by the following first-order conditions.

\[
\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t p_t - \mu_t p_t = 0, \quad (8.9)
\]

\[
\frac{\partial \mathcal{L}}{\partial n^s_t} = -v'(n^s_t) + \mu_t p_t w_t = 0, \quad (8.10)
\]

\[
\frac{\partial \mathcal{L}}{\partial m_{t+1}} = \beta E_t \frac{\partial v}{\partial m_{t+1}} - \mu_t \theta_t = 0, \quad (8.11)
\]

\[
\frac{\partial \mathcal{L}}{\partial b_{t+1}} = \beta E_t \frac{\partial v}{\partial b_{t+1}} - \lambda_t S_t \theta_t - \mu_t S_t \theta_t = 0, \quad (8.12)
\]

\[
\frac{\partial \mathcal{L}}{\partial z_{t+1}} = \beta E_t \frac{\partial v}{\partial z_{t+1}} - \lambda_t p_t q_t - \mu_t p_t q_t = 0. \quad (8.13)
\]

We have the following envelope conditions:

\[
\frac{\partial v}{\partial m_t} = \frac{\partial v}{\partial b_t} = \lambda_t + \mu_t, \quad (8.14)
\]

\[
\frac{\partial v}{\partial z_t} = (\lambda_t + \mu_t) p_t. \quad (8.15)
\]

A binding cash-in-advance constraint implies that \(\lambda_t > 0\). From (8.11), (8.12), and (8.14), we have

\[
\lambda_t = \mu_t (1 - S_t).
\]

Therefore, the cash-in-advance constraint binds if and only if the price of the nominal bond, \(S_t\), is less than one. This implies that the nominal interest rate, \(\frac{1}{S_t} - 1 > 0\).
8.1. A SIMPLE CASH-IN-ADVANCE MODEL WITH PRODUCTION

Now, use (8.14) and (8.15) to substitute for the partial derivatives of the value function in (8.11)-(8.13), and then use (8.9) and (8.10) to substitute for the Lagrange multipliers to obtain

\[ \beta E_t \left( \frac{u'(c_{t+1})}{p_{t+1}} \right) - \theta_t \frac{u'(n_t^s)}{p_t w_t} = 0, \]  

(8.16)

\[ \beta E_t \left( \frac{u'(c_{t+1})}{p_{t+1}} \right) - S_t \frac{u'(c_t)}{p_t} = 0, \]  

(8.17)

\[ \beta E_t u'(c_{t+1}) - q_t u'(c_t) = 0. \]  

(8.18)

Given the definition of \( p_t \), we can write (8.16) and (8.17) more informatively as

\[ \beta E_t \left( \frac{u'(c_{t+1})P_t w_t}{P_{t+1}} \right) = v'(n_t^s) \]  

(8.19)

and

\[ \beta E_t \left( \frac{u'(c_{t+1})}{P_{t+1}} \right) = \frac{S_t u'(c_t)}{P_t}. \]  

(8.20)

Now, equation (8.18) is a familiar pricing equation for a risk free real bond. In equation (8.19), the right-hand side is the marginal disutility of labor, and the left-hand side is the discounted expected marginal utility of labor earnings; i.e. this period’s labor earnings cannot be spent until the following period. Equation (8.20) is a pricing equation for the nominal bond. The right-hand side is the marginal cost, in terms of foregone consumption, from purchasing a nominal bond in period \( t \), and the left-hand side is the expected utility of the payoff on the bond in period \( t + 1 \). Note that the asset pricing relationships, (8.18) and (8.20), play no role in determining the equilibrium.

Profit maximization by the representative firm implies that

\[ w_t = \gamma_t \]  

(8.21)

in equilibrium. Also, in equilibrium the labor market clears,

\[ n_t^s = n_t^d = n_t, \]  

(8.22)

the money market clears, i.e. \( M_t = \bar{M}_t \) or

\[ m_t = 1, \]  

(8.23)
and the bond markets clear,

\[ b_t = z_t = 0. \] (8.24)

Given the equilibrium conditions (8.21)-(8.24), (8.3), (8.4), and (8.8) (with equality), we also have

\[ c_t = \gamma_t n_t. \] (8.25)

Also, (8.21)-(8.24), (8.3), (8.4), and (8.7) (with equality) give

\[ p_t c_t = \theta_t, \]

or, using (8.25),

\[ p_t \gamma_t n_t = \theta_t. \] (8.26)

Now, substituting for \( c_t \) and \( p_t \) in (8.16) using (8.25) and (8.26), we get

\[
\beta E_t \left[ \frac{\gamma_{t+1} n_{t+1} u'(\gamma_{t+1} n_{t+1})}{\theta_{t+1}} \right] - n_t v'(n_t) = 0. \] (8.27)

Here, (8.27) is the stochastic law of motion for employment in equilibrium. This equation can be used to solve for \( n_t \) as a function of the state \((\gamma_t, \theta_t)\). Once \( n_t \) is determined, we can then work backward, to obtain the price level, from (8.26),

\[ P_t = \frac{\theta_t M_t}{\gamma_t n_t}, \] (8.28)

and consumption from (8.25). Note that (8.28) implies that the income velocity of money, defined by

\[ V_t \equiv \frac{P_t y_t}{\theta_t M_t}, \]

is equal to 1. Empirically, the velocity of money is a measure of the intensity with which the stock of money is used in exchange, and there are regularities in the behavior of velocity over the business cycle which we would like our models to explain. In this and other cash-in-advance models, the velocity of money is fixed if the cash-in-advance constraint
binds, as the stock of money turns over once per period. This can be viewed as a defect of this model.

Substituting for \( p_t \) and \( c_t \) in the asset pricing relationships (8.17) and (8.18) using (8.25) and (8.26) gives

\[
\beta E_t \left[ \frac{\gamma_{t+1}n_{t+1}u'(\gamma_{t+1}n_{t+1})}{\theta_{t+1}} \right] - S_t \gamma_t n_t u'(\gamma_t n_t) = 0, \quad (8.29)
\]

\[
\beta E_t u'(\gamma_{t+1}n_{t+1}) - q_t u'(\gamma_t n_t) = 0. \quad (8.30)
\]

From (8.28) and (8.29), we can also obtain a simple expression for the price of the nominal bond,

\[
S_t = \frac{v'(n_t)}{\gamma_t u'(\gamma_t n_t)}. \quad (8.31)
\]

Note that, for our maintained assumption of a binding cash-in-advance constraint to be correct, we require that \( S_t < 1 \), or that the equilibrium solution satisfy

\[
v'(n_t) < \gamma_t u'(\gamma_t n_t). \quad (8.32)
\]

## 8.2 Examples

### 8.2.1 Certainty

Suppose that \( \gamma_t = \gamma \) and \( \theta_t = \theta \) for all \( t \), where \( \gamma \) and \( \theta \) are positive constants, i.e. there are no technology shocks, and the money supply grows at a constant rate. Then, \( n_t = n \) for all \( t \), where, from (8.27), \( n \) is the solution to

\[
\frac{\beta \gamma u'(\gamma n)}{\theta} - v'(n) = 0. \quad (8.33)
\]

Now, note that, for the cash-in-advance constraint to bind, from (8.32) we must have

\[
\theta > \beta,
\]

that is the money growth factor must be greater than the discount factor. From (8.28) and (8.4), the price level is given by

\[
P_t = \frac{\theta^{t+1} \bar{M}_0}{\gamma n}, \quad (8.34)
\]
and the inflation rate is
\[ \pi_t = \frac{P_{t+1}}{P_t} - 1 = \theta - 1. \] (8.35)

Here, money is neutral in the sense that changing the level of the money supply, i.e. changing \( \bar{M}_0 \), has no effect on any real variables, but only increases all prices in proportion (see 8.34). Note that \( \bar{M}_0 \) does not enter into the determination of \( n \) (which determines output and consumption) in (8.33). However, if the monetary authority changes the rate of growth of the money supply, i.e. if \( \theta \) increases, then this does have real effects; money is not super-neutral in this model. Comparative statics in equation (8.33) gives
\[ \frac{dn}{d\theta} = \frac{\beta \gamma u'(\gamma n)}{\theta \beta \gamma^2 u''(\gamma n) - \theta^2 v''(n)} < 0. \]

Note also that, from (8.35), an increase in the money growth rate implies a one-for-one increase in the inflation rate. From (8.34), there is a level effect on the price level of a change in \( \theta \), due to the change in \( n \), and a direct growth rate effect through the change in \( \theta \). Employment, output, and consumption decrease with the increase in the money growth rate through a labor supply effect. That is, an increase in the money growth rate causes an increase in the inflation rate, which effectively acts like a tax on labor earnings. Labor earnings are paid in cash, which cannot be spent until the following period, and in the intervening time purchasing power is eroded. With a higher inflation rate, the representative agent’s real wage falls, and he/she substitutes leisure for labor.

With regard to asset prices, from (8.29) and (8.30) we get
\[ q_t = \beta \]
and
\[ S_t = \frac{\beta}{\theta} \]

The real interest rate is given by
\[ r_t = \frac{1}{q_t} - 1 = \frac{1}{\beta} - 1, \]
i.e. the real interest rate is equal to the discount rate, and the nominal interest rate is
\[ R_t = \frac{1}{S_t} - 1 = \frac{\theta}{\beta} - 1 \]
Therefore, we have
\[ R_t - r_t = \frac{\theta - 1}{\beta} \cong \theta - 1 = \pi_t, \] (8.36)
which is a good approximation if \( \beta \) is close to 1. Here, (8.36) is a Fisher relationship, that is the difference between the nominal interest rate and the real interest rate is approximately equal to the inflation rate. Increases in the inflation rate caused by increases in money growth are reflected in an approximately one-for-one increase in the nominal interest rate, with no effect on the real rate.

8.2.2 Uncertainty

Now, suppose that \( \theta_t \) and \( \gamma_t \) are each i.i.d. random variables. Then, there exists a competitive equilibrium where \( n_t \) is also i.i.d., and (8.29) gives
\[ \beta \psi - n_t \psi'(n_t) = 0, \] (8.37)
where \( \psi \) is a constant. Then, (8.37) implies that \( n_t = n \), where \( n \) is a constant. From (8.29) and (8.30), we obtain
\[ \beta \psi - S_t \gamma_t n u' \gamma_t n = 0 \] (8.38)
and
\[ \beta \omega - q_t \omega' \gamma_t n = 0, \] (8.39)
where \( \omega \) is a constant. Note in (8.37)-(8.39) that \( \theta_t \) has no effect on output, employment, consumption, or real and nominal interest rates. In this model, monetary policy has no effect except to the extent that it is anticipated. Here, given that \( \theta_t \) is i.i.d., the current money growth rate provides no information about future money growth, and so there are no real effects. Note however that the probability distribution for \( \theta_t \) is important in determining the equilibrium, as this well in general affect \( \psi \) and \( \omega \).
The technology shock, $\gamma_t$, will have real effects here. Since $y_t = \gamma_t n$, high $\gamma_t$ implies high output and consumption. From (8.39), the increase in output results in a decrease in the marginal utility of consumption, and $q_t$ rises (the real interest rate falls) as the representative consumer attempts to smooth consumption into the future. From (8.28), the increase in output causes a decrease in the price level, $P_t$, so that consumers expect higher inflation. The effect on the nominal interest rate, from (8.38), is ambiguous. Comparative statics gives

$$\frac{dS_t}{d\gamma_t} = -\frac{S_t}{\gamma_t} \left[ \frac{\gamma_t u'(\gamma_t n)}{u''(\gamma_t n)} + 1 \right]$$

Therefore, if the coefficient of relative risk aversion is greater than one, $S_t$ rises (the nominal interest rate falls); otherwise the nominal interest rate rises. There are two effects on the nominal interest rate. First, the nominal interest rate will tend to fall due to the same forces that cause the real interest rate to fall. That is, consumers buy nominal bonds in order to consume more in the future as well as today, and this pushes up the price of nominal bonds, reducing the nominal interest rate. Second, there is a positive anticipated inflation effect on the nominal interest rate, as inflation is expected to be higher. Which effect dominates depends on the strength of the consumption-smoothing effect, which increases as curvature in the utility function increases.

### 8.3 Optimality

In this section we let $\gamma_t = \gamma$, a constant, for all $t$, and allow $\theta_t$ to be determined at the discretion of the monetary authority. Suppose that the monetary authority chooses an optimal money growth policy $\theta^*_t$ so as to maximize the welfare of the representative consumer. We want to determine the properties of this optimal growth rule. To do so, first consider the social planner’s problem in the absence of monetary arrangements. The social planner solves

$$\max_{\{n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(\gamma n_t) - v(n_t)],$$
8.4. PROBLEMS WITH THE CASH-IN-ADVANCE MODEL

but this breaks down into a series of static problems. Letting \( n^*_t \) denote the optimal choice for \( n_t \), the optimum is characterized by the first-order condition

\[
\gamma u'(\gamma n^*_t) - v'(n^*_t) = 0,
\]

and this then implies that \( n^*_t = n^* \), a constant, for all \( t \). Now, we want to determine the \( \theta^*_t \) which will imply that \( n_t = n^* \) is a competitive equilibrium outcome for this economy. From (8.27), we therefore require that

\[
\beta E_t \left[ \frac{\gamma n^* u'(\gamma n^*)}{\theta^*_{t+1}} \right] - n^* v'(n^*) = 0,
\]

and from (8.40) this requires that

\[
\theta^*_{t+1} = \beta,
\]

i.e., the money supply decreases at the discount rate. The optimal money growth rule in (8.41) is referred to as a “Friedman rule” (see Friedman 1969) or a “Chicago rule.” Note that this optimal money rule implies, from (8.29), that \( S_t = 1 \) for all \( t \), i.e. the nominal interest rate is zero and the cash-in-advance constraint does not bind. In this model, a binding cash-in-advance constraint represents an inefficiency, as does a positive nominal interest rate. If alternative assets bear a higher real return than money, then the representative consumer economizes too much on money balances relative to the optimum. Producing a deflation at the optimal rate (the rate of time preference) eliminates the distortion of the labor supply decision and brings about an optimal allocation of resources.

8.4 Problems With the Cash-in-Advance Model

While this model gives some insight into the relationship between money, interest rates, and real activity, in the long run and over the business cycle, the model has some problems in its ability to fit the facts. The first problem is that the velocity of money is fixed in this model, but is highly variable in the data. There are at least two straightforward
means for curing this problem (at least in theory). The first is to define preferences over “cash goods” and “credit goods” as in Lucas and Stokey (1987). Here, cash goods are goods that are subject to the cash-in-advance constraint. In this context, variability in inflation causes substitution between cash goods and credit goods, which in turn leads to variability in velocity. A second approach is to change some of the timing assumptions concerning transactions in the model. For example, Svensson (1985) assumes that the asset market opens before the current money shock is known. Thus, the cash-in-advance constraint binds in some states of the world but not in others, and velocity is variable. However, neither of these approaches works empirically; Hodrick, Kocherlakota, and Lucas (1991) show that these models do not produce enough variability in velocity to match the data.

Another problem is that, in versions of this type of model where money growth is serially correlated (as in practice), counterfactual responses to surprise increases in money growth are predicted. Empirically, money growth rates are positively serially correlated. Given this, if there is high money growth today, high money growth is expected tomorrow. But this will imply (in this model) that labor supply falls, output falls, and, given anticipated inflation, the nominal interest rate rises. Empirically, surprise increases in money growth appear to generate short run increases in output and employment, and a short run decrease in the nominal interest rate. Work by Lucas (1990) and Fuerst (1992) on a class of “liquidity effect” models, which are versions of the cash-in-advance approach, can obtain the correct qualitative responses of interest rates and output to money injections.

A third problem has to do with the lack of explicitness in the basic approach to modeling monetary arrangements here. The model is silent on what the objects are which enter the cash-in-advance constraint. Implicit in the model is the assumption that private agents cannot produce whatever it is that satisfies cash-in-advance. If they could, then there could not be an equilibrium with a positive nominal interest rate, as a positive nominal interest rate represents a profit opportunity for private issuers of money substitutes. Because the model is not explicit about the underlying restrictions which support cash-in-advance, and because it requires the modeler to define at the outset what money is, the cash-in-advance approach is virtually useless for studying sub-
stitution among money substitutes and the operation of the banking system. There are approaches which model monetary arrangements at a deeper level, such as in the overlapping generations model (Wallace 1980) or in search environments (Kiyotaki and Wright 1989), but these approaches are not easily amenable to empirical application.

A last problem has to do with the appropriateness of using a cash-in-advance model for studying quarterly (or even monthly) fluctuations in output, prices, and interest rates. Clearly, it is very difficult to argue that consumption expenditures during the current quarter (or month) are constrained by cash acquired in the previous quarter (or month), given the low cost of visiting a cash machine or using a credit card.

8.5 References


