Chapter 6

Consumption and Asset Pricing

In this chapter we will examine the theory of consumption behavior and asset pricing in dynamic representative agent models. These two topics are treated together because there is a close relationship between the behavior of consumption and asset prices in this class of models. That is, consumption theory typically treats asset prices as being exogenous and determines optimal consumption-savings decisions for a consumer. However, asset pricing theory typically treats aggregate consumption as exogenous while determining equilibrium asset prices. The stochastic implications of consumption theory and asset pricing theory, captured in the stochastic Euler equations from the representative consumer’s problem, look quite similar.

6.1 Consumption

The main feature of the data that consumption theory aims to explain is that aggregate consumption is smooth, relative to aggregate income. Traditional theories of consumption which explain this fact are Friedman’s permanent income hypothesis and the life cycle hypothesis of Modigliani and Brumberg. Friedman’s and Modigliani and Brumberg’s ideas can all be expositied in a rigorous way in the context of the class of representative agent models we have been examining.
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6.1.1 Consumption Behavior Under Certainty

The model we introduce here captures the essentials of consumption-smoothing behavior which are important in explaining why consumption is smoother than income. Consider a consumer with initial assets $A_0$ and preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, $c_t$ is consumption, and $u(\cdot)$ is increasing, strictly concave, and twice differentiable. The consumer’s budget constraint is

$$A_{t+1} = (1 + r)(A_t - c_t + w_t),$$

for $t = 0, 1, 2, \ldots$, where $r$ is the one-period interest rate (assumed constant over time) and $w_t$ is income in period $t$, where income is exogenous. We also assume the no-Ponzi-scheme condition

$$\lim_{t \to \infty} \frac{A_t}{(1 + r)^t} = 0.$$

This condition and (6.2) gives the intertemporal budget constraint for the consumer,

$$\sum_{t=0}^{\infty} \frac{c_t}{(1 + r)^t} = A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1 + r)^t}$$

The consumer’s problem is to choose $\{c_t, A_{t+1}\}_{t=0}^{\infty}$ to maximize (6.1) subject to (6.2). Formulating this problem as a dynamic program, with the value function $v(A_t)$ assumed to be concave and differentiable, the Bellman equation is

$$v(A_t) = \max_{A_{t+1}} \left[ u \left( w_t + A_t - \frac{A_{t+1}}{1 + r} \right) + \beta v(A_{t+1}) \right].$$

The first-order condition for the optimization problem on the right-hand side of the Bellman equation is

$$-\frac{1}{1 + r} u' \left( w_t + A_t - \frac{A_{t+1}}{1 + r} \right) + \beta' v'(A_{t+1}) = 0,$$

and the envelope theorem gives

$$v'(A_t) = u' \left( w_t + A_t - \frac{A_{t+1}}{1 + r} \right).$$
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Therefore, substituting in (6.4) using (6.5) and (6.2) gives

\[
\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r \tag{6.6}
\]

That is, the intertemporal marginal rate of substitution is equal to one plus the interest rate at the optimum.

Now, consider some special cases. If \(1 + r = \frac{1}{\beta}\), i.e. if the interest rate is equal to the discount rate, then (6.6) gives

\[c_t = c_{t+1} = c\]

for all \(t\), where, from (6.3), we get

\[c = \left(\frac{r}{1 + r}\right) \left(A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1 + r)^t}\right). \tag{6.7}\]

Here, consumption in each period is just a constant fraction of discounted lifetime wealth or “permanent income.” The income stream given by \(\{w_t\}_{t=0}^{\infty}\) could be highly variable, but the consumer is able to smooth consumption perfectly by borrowing and lending in a perfect capital market. Also, note that (6.7) implies that the response of consumption to an increase in permanent income is very small. That is, suppose a period is a quarter, and take \(r = .01\) (an interest rate of approximately 4% per annum). Then (6.7) implies that a $1 increase in current income gives an increase in current consumption of $.0099. This is an important implication of the permanent income hypothesis: because consumers smooth consumption over time, the impact on consumption of a temporary increase in income is very small.

Another example permits the discount factor to be different from the interest rate, but assumes a particular utility function, in this case

\[u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha},\]

where \(\gamma > 0\). Now, from (6.6) we get

\[\frac{c_{t+1}}{c_t} = [\beta(1 + r)]^{\frac{1}{\alpha}}, \tag{6.8}\]
so that consumption grows at a constant rate for all $t$. Again, the consumption path is smooth. From (6.8) we have

$$c_t = c_0 \left[ \beta (1 + r) \right]^{\frac{t}{\alpha}},$$

and solving for $c_0$ using (6.3), we obtain

$$c_0 = \left[ 1 - \beta \frac{1}{\alpha} (1 + r) \right]^{\frac{1}{\alpha}} \left[ A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1 + r)^t} \right].$$

6.1.2 Consumption Behavior Under Uncertainty

Friedman’s permanent income hypothesis was a stochastic theory, aimed at explaining the regularities in short run and long run consumption behavior, but Friedman did not develop his theory in the context of an optimizing model with uncertainty. This was later done by Hall (1978), and the following is essentially Hall’s model.

Consider a consumer with preferences given by

$$E_0 \sum \beta^t u(c_t),$$

where $u(\cdot)$ has the same properties as in the previous section. The consumer’s budget constraint is given by (6.2), but now the consumer’s income, $w_t$, is a random variable which becomes known at the beginning of period $t$. Given a value function $v(A_t, w_t)$ for the consumer’s problem, the Bellman equation associated with the consumer’s problem is

$$v(A_t, w_t) = \max_{A_{t+1}} \left[ u(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E_t v(A_{t+1}, w_{t+1}) \right],$$

and the first-order condition for the maximization problem on the right-hand side of the Bellman equation is

$$-\frac{1}{1+r} u'(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E_t v_1(A_{t+1}, w_{t+1}). \quad (6.9)$$

We also have the following envelope condition:

$$v_1(A_t, w_t) = u'(A_t + w_t - \frac{A_{t+1}}{1+r}). \quad (6.10)$$
Therefore, from (6.2), (6.9), and (6.10), we obtain

$$E_t u'(c_{t+1}) = \frac{1}{\beta(1+r)} u'(c_t). \quad (6.11)$$

Here, (6.11) is a stochastic Euler equation which captures the stochastic implications of the permanent income hypothesis for consumption. Essentially, (6.11) states that $u'(c_t)$ is a martingale with drift. However, without knowing the utility function, this does not tell us much about the path for consumption. If we suppose that $u(\cdot)$ is quadratic, i.e. $u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2$, where $\bar{c} > 0$ is a constant, (6.11) gives

$$E_t c_t = \left[ \frac{\beta(1+r) - 1}{\beta(1+r)} \right] c_t,$$

so that consumption is a martingale with drift. That is, consumption is smooth in the sense that the only information required to predict future consumption is current consumption. A large body of empirical work (summarized in Hall 1989) comes to the conclusion that (6.11) does not fit the data well. Basically, the problem is that consumption is too variable in the data relative to what the theory predicts; in practice, consumers respond more strongly to changes in current income than theory predicts they should.

There are at least two explanations for the inability of the permanent income model to fit the data. The first is that much of the work on testing the permanent income hypothesis is done using aggregate data. But in the aggregate, the ability of consumers to smooth consumption is limited by the investment technology. In a real business cycle model, for example, asset prices move in such a way as to induce the representative consumer to consume what is produced in the current period. That is, interest rates are not exogenous (or constant, as in Hall’s model) in general equilibrium. In a real business cycle model, the representative consumer has an incentive to smooth consumption, and these models fit the properties of aggregate consumption well.

A second possible explanation, which has been explored by many authors (see Hall 1989), is that capital markets are imperfect in practice. That is, the interest rates at which consumers can borrow are typically much higher than the interest rates at which they can lend,
and sometimes consumers cannot borrow on any terms. This limits the ability of consumers to smooth consumption, and makes consumption more sensitive to changes in current income.

6.2 Asset Pricing

In this section we will study a model of asset prices, developed by Lucas (1978), which treats consumption as being exogenous, and asset prices as endogenous. This asset pricing model is sometimes referred to as the ICAPM (intertemporal capital asset pricing model) or the consumption-based capital asset pricing model.

This is a representative agent economy where the representative consumer has preferences given by

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \]  

(6.12)

where \( 0 < \beta < 1 \) and \( u(\cdot) \) is strictly increasing, strictly concave, and twice differentiable. Output is produced on \( n \) productive units, where \( y_{it} \) is the quantity of output produced on productive unit \( i \) in period \( t \). Here \( y_{it} \) is random. We can think of each productive unit as a fruit tree, which drops a random amount of fruit each period.

It is clear that the equilibrium quantities in this model are simply

\[ c_t = \sum_{i=1}^{n} y_{it}, \]  

(6.13)

but our interest here is in determining competitive equilibrium prices. However, what prices are depends on the market structure. We will suppose an stock market economy, where the representative consumer receives an endowment of 1 share in each productive unit at \( t = 0 \), and the stock of shares remains constant over time. Each period, the output on each productive unit (the dividend) is distributed to the shareholders in proportion to their share holdings, and then shares are traded on competitive markets. Letting \( p_{it} \) denote the price of a share in productive unit \( i \) in terms of the consumption good, and \( z_{it} \) the
quantity of shares in productive unit \( i \) held at the beginning of period \( t \), the representative consumer’s budget constraint is given by

\[
\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t = \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}), \quad (6.14)
\]

for \( t = 0, 1, 2, \ldots \). The consumer’s problem is to maximize (6.12) subject to (6.14). Letting \( p_t, z_t, \) and \( y_t \) denote the price vector, the vector of share holdings, and the output vector, for example \( y_t = (y_{1t}, y_{2t}, \ldots, y_{nt}) \), we can specify a value function for the consumer \( v(z_t, p_t, y_t) \), and write the Bellman equation associated with the consumer’s problem as

\[
v(z_t, p_t, y_t) = \max_{z_{it+1}} \left[ u(c_t) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right]
\]

subject to (6.14). Lucas (1978) shows that the value function is differentiable and concave, and we can substitute using (6.14) in the objective function to obtain

\[
v(z_t, p_t, y_t) = \max_{z_{it+1}} \left\{ u \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - p_{it} z_{i,t+1} \right) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right\}.
\]

Now, the first-order conditions for the optimization problem on the right-hand side of the above Bellman equation are

\[
-p_{it} u' \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - p_{it} z_{i,t+1} \right) + \beta E_t \frac{\partial v}{\partial z_{i,t+1}} = 0, \quad (6.15)
\]

for \( i = 1, 2, \ldots, n \). We have the following envelope conditions:

\[
\frac{\partial v}{\partial z_{it}} = (p_{it} + y_{it}) u' \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - p_{it} z_{i,t+1} \right) \quad (6.16)
\]

Substituting in (6.15) using (6.13), (6.14), and (6.16) then gives

\[
-p_{it} u' \left( \sum_{i=1}^{n} y_{it} \right) + \beta E_t \left[ (p_{i,t+1} + y_{i,t+1}) u' \left( \sum_{i=1}^{n} y_{i,t+1} \right) \right] = 0, \quad (6.17)
\]

for \( i = 1, 2, \ldots, n \), or

\[
p_{it} = E_t \left[ (p_{i,t+1} + y_{i,t+1}) \frac{\beta u' (c_{t+1})}{u' (c_t)} \right] = 0. \quad (6.18)
\]
That is, the current price of a share is equal to the expectation of the product of the future payoff on that share with the intertemporal marginal rate of substitution. Perhaps more revealing is to let $\pi_{it}$ denote the gross rate of return on share $i$ between period $t$ and period $t+1$, i.e.

$$
\pi_{it} = \frac{p_{i,t+1} + y_{i,t+1}}{p_{it}},
$$

and let $m_t$ denote the intertemporal marginal rate of substitution,

$$
m_t = \beta u'(c_{t+1}) \frac{u'(c_t)}{u'(c_t)}.
$$

Then, we can rewrite equation (6.18) as

$$
E_t (\pi_{it} m_t) = 1,
$$
or, using the fact that, for any two random variables, $X$ and $Y$, $cov(X, Y) = E(XY) - E(X)E(Y)$,

$$
cov_t(\pi_{it}, m_t) + E_t (\pi_{it}) E_t (m_t) = 1.
$$

Therefore, shares with high expected returns are those for which the covariance of the asset’s return with the intertemporal marginal rate of substitution is low. That is, the representative consumer will pay a high price for an asset which is likely to have high payoffs when aggregate consumption is low. We can also rewrite (6.18), using repeated substitution and the law of iterated expectations (which states that, for a random variable $x_t$, $E_t \left[ E_{t+s} x_{t+s} \right] = E_t x_{t+s}$, $s' \geq s \geq 0$), to get

$$
p_{it} = E_t \left[ \sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'(c_s)}{u'(c_t)} y_{i,s} \right].
$$

That is, we can write the current share price for any asset as the expected present discounted value of future dividends, where the discount factors are intertemporal marginal rates of substitution. Note here that the discount factor is not constant, but varies over time since consumption is variable.
Examples

Equation (6.17) can be used to solve for prices, and we will show here how this can be done in some special cases.

First, suppose that \( y_t \) is an i.i.d. random variable. Then, it must also be true that \( p_t \) is i.i.d. This then implies that

\[
E_t \left[ (p_{i,t+1} + y_{i,t+1}) u' \left( \sum_{i=1}^{n} y_{i,t+1} \right) \right] = A_i, \tag{6.20}
\]

for \( i = 1, 2, ..., n \), where \( A_i > 0 \) is a constant. That is, the expression inside the expectation operator in (6.20) is a function of \( p_{i,t+1} \) and \( y_{i,t+1} \), each of which is unpredictable given information in period \( t \), therefore the function is unpredictable given information in period \( t \). Given (6.17) and (6.20), we get

\[
p_{it} = \frac{\beta A_i}{u' \left( \sum_{i=1}^{n} y_{it} \right)}
\]

Therefore, if aggregate output (which is equal to aggregate consumption here) is high, then the marginal utility of consumption is low, and the current price of the asset is high. That is, if current dividends on assets are high, the representative consumer will want to consume more today, but will also wish to save by buying more assets so as to smooth consumption. However, in the aggregate, the representative consumer must be induced to consume aggregate output (or equivalently, to hold the supply of available assets), and so asset prices must rise.

A second special case is where there is risk neutrality, that is \( u(c) = c \). From (6.17), we then have

\[
p_{it} = \beta E_t(p_{i,t+1} + y_{i,t+1}),
\]

i.e. the current price is the discount value of the expected price plus the dividend for next period, or

\[
E_t \left[ \frac{p_{i,t+1} + y_{i,t+1} - p_{it}}{p_{it}} \right] = \frac{1}{\beta} - 1. \tag{6.21}
\]

Equation (6.21) states that the rate of return on each asset is unpredictable given current information, which is sometimes taken in the
Finance literature as an implication of the “efficient markets hypothesis.” Note here, however, that (6.21) holds only in the case where the representative consumer is risk neutral. Also, (6.19) gives

\[ p_t = E_t \sum_{s=t+1}^{\infty} \beta^{s-t} y_{i,s}, \]

or the current price is the expected present discounted value of dividends.

A third example considers the case where \( u(c) = \ln c \) and \( n = 1 \); that is, there is only one asset, which is simply a share in aggregate output. Also, we will suppose that output takes on only two values, \( y_t = y_1, y_2 \), with \( y_1 > y_2 \), and that \( y_t \) is i.i.d. with \( \Pr[y_t = y_1] = \pi \), \( 0 < \pi < 1 \). Let \( p_i \) denote the price of a share when \( y_t = y_i \) for \( i = 1, 2 \). Then, from (6.17), we obtain two equations which solve for \( p_1 \) and \( p_2 \),

\[
\begin{align*}
p_1 &= \beta \left[ \pi \frac{y_1}{y_1} (p_1 + y_1) + (1 - \pi) \frac{y_1}{y_2} (p_2 + y_2) \right] \\
p_2 &= \beta \left[ \pi \frac{y_2}{y_1} (p_1 + y_1) + (1 - \pi) \frac{y_2}{y_2} (p_2 + y_2) \right]
\end{align*}
\]

Since the above two equations are linear in \( p_1 \) and \( p_2 \), it is straightforward to solve, obtaining

\[
\begin{align*}
p_1 &= \frac{\beta y_1}{1 - \beta} \\
p_2 &= \frac{\beta y_2}{1 - \beta}
\end{align*}
\]

Note here that \( p_1 > p_2 \), that is the price of the asset is high in the state when aggregate output is high.

**Alternative Assets and the “Equity Premium Puzzle”**

Since this is a representative agent model (implying that there can be no trade in equilibrium) and because output and consumption are exogenous, it is straightforward to price a wide variety of assets in this type of model. For example, suppose we allow the representative agent to borrow and lend. That is, there is a risk-free asset which trades on
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a competitive market at each date. This is a one-period risk-free bond which is a promise to pay one unit of consumption in the following period. Let \( b_{t+1} \) denote the quantity of risk-free bonds acquired in period \( t \) by the representative agent (note that \( b_{t+1} \) can be negative; the representative agent can issue bonds), and let \( q_t \) denote the price of a bond in terms of the consumption good in period \( t \). The representative agent’s budget constraint is then

\[
\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t + q_t b_{t+1} = \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) + b_t
\]

In equilibrium, we will have \( b_t = 0 \), i.e. there is a zero net supply of bonds, and prices need to be such that the bond market clears.

We wish to determine \( q_t \), and this can be done by re-solving the consumer’s problem, but it is more straightforward to simply use equation (6.17), setting \( p_{i,t+1} = 0 \) (since these are one-period bonds, they have no value at the end of period \( t+1 \)) and \( y_{i,t+1} = 1 \) to get

\[
q_t = \beta E_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right]. \tag{6.22}
\]

The one-period risk-free interest rate is then

\[
r_t = \frac{1}{q_t} - 1. \tag{6.23}
\]

If the representative agent is risk neutral, then \( q_t = \beta \) and \( r_t = \frac{1}{\beta} - 1 \), that is the interest rate is equal to the discount rate.

Mehra and Prescott (1985) consider a version of the above model where \( n = 1 \) and there are two assets; an equity share which is a claim to aggregate output, and a one-period risk-free asset as discussed above. They consider preferences of the form

\[
u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}.
\]

In the data set which Mehra and Prescott examine, which includes annual data on risk-free interest rates and the rate of return implied by aggregate dividends and a stock price index, the average rate of return
on equity is approximately 6% higher than the average rate of return on risk-free debt. That is, the average equity premium is about 6%. Mehra and Prescott show that this equity premium cannot be accounted for by Lucas’s asset pricing model.

Mehra and Prescott construct a version of Lucas’s model which incorporates consumption growth, but we will illustrate their ideas here in a model where consumption does not grow over time. The Mehra-Prescott argument goes as follows. Suppose that output can take on two values, $y_1$ and $y_2$, with $y_1 > y_2$. Further, suppose that $y_t$ follows a two-state Markov process, that is

$$
\Pr[y_{t+1} = y_j \mid y_t = y_i] = \pi_{ij}.
$$

We will assume that $\pi_{ii} = \rho$, for $i = 1, 2$, where $0 < \rho < 1$. Here, we want to solve for the asset prices $q_i$, $p_i$, $i = 1, 2$, where $q_i = q_i$ and $p_t = p_i$ when $y_t = y_i$, for $i = 1, 2$. From (6.17), we have

$$
p_1y_1^{-\gamma} = \beta \left[ \rho(p_1 + y_1)y_1^{-\gamma} + (1 - \rho)(p_2 + y_2)y_2^{-\gamma} \right],
$$

$$
p_2y_2^{-\gamma} = \beta \left[ \rho(p_2 + y_2)y_2^{-\gamma} + (1 - \rho)(p_1 + y_1)y_1^{-\gamma} \right].
$$

Also, (6.22) implies that

$$
q_1 = \beta \left[ \rho + (1 - \rho) \left( \frac{y_1}{y_2} \right)^\gamma \right],
$$

$$
q_2 = \beta \left[ \rho + (1 - \rho) \left( \frac{y_2}{y_1} \right)^\gamma \right].
$$

Now, (6.24) and (6.25) are two linear equations in the two unknowns $p_1$ and $p_2$, so (6.24)-(6.27) give us solutions to the four asset prices. Now, to determine risk premia, we first need to determine expected returns. In any period, $t$, the return on the risk-free asset is certain, and given by $r_t$ in (6.23). Let $r_t = r_i$ when $y_t = y_i$ for $i = 1, 2$. For the equity share, the expected return, denoted $R_t$, is given by

$$
R_t = E_t \left( \frac{p_{t+1} + y_{t+1} - p_t}{p_t} \right).
$$
Therefore, letting $R_i$ denote the expected rate of return on the equity share when $y_t = y_i$, we get

$$R_1 = \rho \left( \frac{p_1 + y_1}{p_1} \right) + (1 - \rho) \left( \frac{p_2 + y_2}{p_1} \right) - 1$$

$$R_2 = \rho \left( \frac{p_2 + y_2}{p_2} \right) + (1 - \rho) \left( \frac{p_1 + y_1}{p_2} \right) - 1$$

Now, what we are interested in is the average equity premium that would be observed in the data produced by this model over a long period of time. Given the transition probabilities between output states, the unconditional (long-run) probability of being in either state is $\frac{1}{2}$ here. Therefore, the average equity premium is

$$e(\beta, \gamma, \rho, y_1, y_2) = \frac{1}{2} (R_1 - r_1) + \frac{1}{2} (R_2 - r_2).$$

Mehra and Prescott’s approach is to set $\rho, y_1, \text{and } y_2$ so as to replicate the observed properties of aggregate consumption (in terms of serial correlation and variability), then to find parameters $\beta$ and $\gamma$ such that $e(\beta, \gamma, \rho, y_1, y_2) \approx .06$. What they find is that $\gamma$ must be very large, and much outside of the range of estimates for this parameter which have been obtained in other empirical work.

To understand these results, it helps to highlight the roles played by $\gamma$ in this model. First, $\gamma$ determines the intertemporal elasticity of substitution, which is critical in determining the risk-free rate of interest, $r_t$. That is, the higher is $\gamma$, the lower is the intertemporal elasticity of substitution, and the greater is the tendency of the representative consumer to smooth consumption over time. Thus, a higher $\gamma$ tends to cause an increase in the average risk-free interest rate. Second, the value of $\gamma$ captures risk aversion, which is a primary determinant of the expected return on equity. That is, the higher is $\gamma$ the larger is the expected return on equity, as the representative agent must be compensated more for bearing risk. The problem in terms of fitting the model is that there is not enough variability in aggregate consumption to produce a large enough risk premium, given plausible levels of risk aversion.
6.3 References


