Lag Operators:

\[ LX_t = X_{t-1}, \quad L^n X_t = X_{t-n}, \quad L^{-n} = X_{t+n}, \quad L^0 = 1 \]

Lag Polynomials:

\[ A(L) = a_0 + a_1 L + a_2 L^2 + \ldots = \sum_{j=0}^{\infty} a_j L^j \]
Rational Polynomials:

\[ A(L) = \frac{B(L)}{C(L)}, \quad B(L) = \sum_{j=0}^{\infty} b_j L^j, \quad C(L) = \sum_{j=0}^{\infty} c_j L^j \]

Illustration: \[ A(L) = \frac{1}{1-\lambda L} = \sum_{i=0}^{\infty} (\lambda L)^i \]

Bounded Sequences:

\[ |\lambda| < 1: \quad \frac{1}{1-\lambda L} x_t = \sum_{j=0}^{\infty} \lambda^j x_{t-j} \]

\[ |\lambda| > 1: \quad \frac{1}{1-\lambda L} x_t = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}} x_t = -\sum_{j=0}^{\infty} \lambda^{-j} x_{t+j} \]
Solving Difference Equations:

\[ Y_t = a + \lambda Y_{t-1} + bX_t \]

\[ \Rightarrow (1 - \lambda L)Y_t = a + bX_t \]

\[ \Rightarrow Y_t = \frac{a}{1-\lambda L} + \frac{b}{1-\lambda L}X_t + c\lambda^t \]

\[ \Rightarrow Y_t = \frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{t-i} + c\lambda^t, |\lambda| < 1 \]

Initial Condition: \[ Y_0 = \bar{Y} \]

\[ Y_t = \frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{t-i} + \left( \bar{Y} - \frac{a}{1-\lambda} \right)\lambda^t, |\lambda| < 1 \]
Forward Solution:

\[ Y_t = \frac{a}{1-\lambda} - b \sum_{i=0}^{\infty} \lambda^{-(1+i)} X_{t+i+1} + c\lambda^t, |\lambda| > 1 \]

Second Order Difference Equations:

\[ Y_t = a + t_1 Y_{t-1} + t_2 Y_{t-2} + bX_t \]
\[ (1-t_1L-t_2L^2)Y_t = a + bX_t \]
\[ Y_t = \frac{a}{1-t_1L-t_2L^2} + \frac{b}{1-t_1L-t_2L^2} X_t \]

Factorization:

\[ 1-t_1L-t_2L^2 = (1-\lambda_1L)(1-\lambda_2L) \]
\[(1 - \lambda_1 L)(1 - \lambda_2 L) = 1 - (\lambda_1 + \lambda_2) L + \lambda_1 \lambda_2 L^2\]
\[
\lambda_1 + \lambda_2 = t_1, -\lambda_1 \lambda_2 = t_2
\]

Characteristic Roots:
\[
\lambda = \frac{-t_1 \pm \sqrt{t_1^2 + 4t_2}}{2t_2}
\]

\[(1 - \lambda_1 L)(1 - \lambda_2 L) Y_t = a + b X_t\]
\[
Y_t = \frac{a}{(1 - \lambda_1 L)(1 - \lambda_2 L)} + \frac{b}{(1 - \lambda_1 L)(1 - \lambda_2 L)} X_t + c_1 \lambda_1^t + c_2 \lambda_2^t\]
Example I: Cash Flow Maximization by a Firm

\[
\max \sum_{t=0}^{\infty} \beta^t \left[ P[f(N_t) - C(N_{t+1} - N_t)] - WN_t \right], 0 < \beta < 1
\]

Using quadratic forms gives the following:

\textbf{Euler Equation:} \quad bN_{t+j+1} + \phi N_{t+j} + N_{t+j-1} = d^{-1} \left( w_{t+j} - a_{t+j} - f_0 \right)

\textbf{Definitions:} \quad N = \text{Labor}, \ w = \text{Real Wage}, \\
\quad a = \text{Technology Shock}, \ \phi = \text{Discount Rate}

\textbf{LHS:} \quad (b + \phi L + L^2)N_{t+j+1}
Differential Equations

First Order Equations

\[ \frac{dy}{dt} = w(t) - u(t)y(t), \; y(0) = \bar{y} \]

First Order: first derivative only

Nonhomogenous: \( w(t) \neq 0 \)

Nonautonomous: \( u \) not independent of \( t \)
Solution: a function, $f(t)$, such that $df(t)/dt$ equals the equation above and satisfying the boundary condition

**Homogeneous Equation with Constant Coefficients:**

$$\frac{dy(t)}{dt} = -ay(t) \quad \text{or} \quad \frac{\dot{y}(t)}{y(t)} = -a$$

Solution: $y(t) = Ae^{-at}$

A is an arbitrary constant

satisfying a boundary condition

Verify the Solution:

$$\frac{dy(t)}{dt} = -aAe^{-at} = -ay(t)$$
Determining the Arbitrary Constant:

\[ y(0) = \bar{y} = Ae^{-a_0} = A \]

Complete Solution: \[ y(t) = \bar{y}e^{-at} \]

Nonhomogeneous Equation with Constant Coefficients

\[ \frac{dy(t)}{dt} + ay(t) = b, \quad y(0) = \bar{y} \]

Solution has two parts: the solution to the homogeneous part (the complementary function) and the equilibrium part (the particular integral).
Equilibrium: \[ \frac{dy(t)}{dt} = 0 \Rightarrow ay = b \]

Complete Solution: \[ y(t) = Ae^{-at} + \frac{b}{a} = \left( \frac{y - b}{a} \right)e^{-at} + \frac{b}{a} \]

Homogeneous Case: Nonconstant Coefficients

\[ \dot{y}(t) = -u(t)y(t), \quad \frac{\dot{y}}{y} = -u(t) \]

Integrate both sides to get

\[ \int \frac{\dot{y}}{y} \, dt = -\int u(t) \, dt \]

\[ \ln y + C = -\int u(t) \, dt \]
\[ y(t) = e^{-c} e^{-\int u(t) \, dt}, \quad y(t) = Ae^{-\int u(t) \, dt} \]

If \( u(t) = u \; \forall \; t \quad \Rightarrow \quad \int u(t) \, dt = ut \)

**Nonhomogeneous Equation with Nonconstant Coefficients**

\[ y(t) = e^{-\int u(t) \, dt} \left[ A + \int w(t) e^{\int u(t) \, dt} \, dt \right] \]

**An Illustration: Capital Accumulation**

\[ \dot{K}(t) = I(t) - \delta K(t), \; K(0) = K_0 \]
Nonlinear Differential Equations

\[ \dot{y}(t) = f(y(t)) \]

Explicit analytical solutions are frequently unavailable but qualitative solutions can often be found.

\[ K(t) = K_0 e^{-\delta t} + \int_0^t I(\tau) e^{-\delta(t-\tau)} d\tau \]

\[ K(t) = e^{-\delta t} \left[ K_0 + \int_0^t I(\tau) e^{\delta \tau} d\tau \right] \]
Stability: \[ \frac{\partial \dot{y}(t)}{\partial y(t)} = f'(y(t)) \]

Stable if \( f' < 0 \)
Unstable if \( f' > 0 \)
Higher Order Equations
\[ \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = b \]

Trial Solution:
\[ y(t) = Ae^{-rt} \Rightarrow \dot{y} = rAe^{-rt}, \ddot{y} = r^2 Ae^{-rt} \]

Substituting into the differential equation gives
\[ \left( r^2 + a_1 r + a_2 \right) Ae^{-rt} = 0 \]

Now find the roots of the quadratic
\[ r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]

where \( r_1 + r_2 = -a_1, \ r_1r_2=a_2. \)
Both of the following can be shown to be solutions:

\[ y_1(t) = Ae^{r_1 t}, y_2(t) = Ae^{r_2 t} \]

Therefore the complete solution is

\[ y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{b}{a_2} \]

Roots can be real or complex, repeated or distinct. Since a slight perturbation of underlying parameters induces distinct roots, we will always assume in this course that characteristic roots are distinct.
Boundary Conditions

Suppose that there are two boundary conditions in this problem, \( y(0) \), and \( \frac{dy(0)}{dt} \). Set \( t \) equal to zero in the differential equation to get

\[
y(0) = A_1 + A_2 + \frac{b}{a_2}
\]

Differentiate the solution and set \( t = 0 \) to get

\[
\dot{y}(0) = r_1 A_1 + r_2 A_2
\]

Solve simultaneously to get

\[
A_1 = \frac{r_2 \left[ y(0) - \frac{b}{a_2} \right] - \dot{y}(0)}{r_2 - r_1}, \quad A_2 = \frac{r_1 \left[ y(0) - \frac{b}{a_2} \right] - \dot{y}(0)}{r_1 - r_2}
\]
**Simultaneous Systems:** consider $\ddot{y} + a_1 \dot{y} + a_2 y = 0$

Using the substitution $\dot{y} = x \Rightarrow \ddot{y} = \dot{x}$

gives rise to the simultaneous system

$$\begin{align*}
\dot{x} + a_1 x + a_2 y &= 0 \\
\dot{y} &= x
\end{align*}$$

**Homogeneous System with Constant Coefficients**

$$\begin{align*}
\dot{y} &= a_{11} y + a_{12} z \\
\dot{z} &= a_{21} y + a_{22} z
\end{align*}$$

$$\begin{bmatrix}
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
y \\
z
\end{bmatrix}$$
Try exponential solutions as before, giving

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda t}$$

For this to hold requires

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Nontrivial solutions to this require

$$|a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda| = 0 \Rightarrow \lambda^2 - \text{Tr}[a] \lambda + |a| = 0$$
\[
\text{Tr}[a] = a_{11} + a_{22}, \quad |a| = a_{11}a_{22} - a_{12}^2
\]

If the roots are distinct,

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
a_{11} - \lambda_i & a_{12} \\
a_{21} & a_{22} - \lambda_i
\end{bmatrix} \begin{bmatrix}
\alpha_1^i \\
\alpha_2^i
\end{bmatrix}
\]

Set \( \alpha_1^i = 1 \), \( \Rightarrow \alpha_2^i = \frac{\lambda_i - a_{11}}{a_{12}} \)

Solution:

\[
\begin{bmatrix}
y(t) \\
z(t)
\end{bmatrix} = \begin{bmatrix}
A_1 & A_2 \\
A_1 \alpha_1^1 & A_1 \alpha_2^2
\end{bmatrix} \begin{bmatrix}
e^{-\lambda_1 t} \\
e^{-\lambda_2 t}
\end{bmatrix}
\]
Stability of $N^{th}$ Order System

$$|a - \lambda I| = 0$$

Roots from this $n$th order polynomial must be negative or have negative real parts if they are complex numbers. System will be stable if $[a]$ is negative definite. A necessary condition for this to be true is that $\text{Tr}[a] < 0$.

Linear Approximations:

$$\dot{y} = f(y, z)$$

$$\dot{z} = g(y, z)$$
Taylor Expansions

\[ f(y, z) \approx f(y^*, z^*) + f_y(y^*, z^*)(y - y^*) + f_z(y^*, z^*)(z - z^*) \]

Neglect higher order terms.

What are \( y^* \) and \( z^* \)?

They arise from \( f(y^*, z^*) = 0 = g(y^*, z^*) \)

Define the new variables: \( \tilde{y}(t) = y(t) - y^*, \tilde{z}(t) = z(t) - z^* \)

Approximate Linear System:

\[
\begin{bmatrix}
\dot{\tilde{y}}(t) \\
\dot{\tilde{z}}(t)
\end{bmatrix} =
\begin{bmatrix}
f_y(y^*, z^*) & f_z(y^*, z^*) \\
g_y(y^*, z^*) & g_z(y^*, z^*)
\end{bmatrix}
\begin{bmatrix}
\tilde{y}(t) \\
\tilde{z}(t)
\end{bmatrix}
\]
Phase Diagrams:

-provides qualitative information about the solution paths of nonlinear systems.

Nonlinear System: \( \dot{x} = f(x, y), \dot{y} = g(x, y) \)

Singular Curves: \( f(x^*, y^*) = g(x^*, y^*) = 0 \)

Establish the slopes of the singular curves by totally differentiating the singular curves.

\[
\begin{align*}
\frac{\partial y}{\partial x} \bigg|_{\dot{x}=0} &= -\frac{f_x}{f_y} > 0 \text{ say } & \frac{\partial y}{\partial x} \bigg|_{\dot{y}=0} &= -\frac{g_x}{g_y} < 0 \text{ say }
\end{align*}
\]
Now establish the directions of motion. Suppose that

\[
\frac{\partial \dot{x}}{\partial x} = f_x < 0, \quad \frac{\partial \dot{y}}{\partial y} = g_y < 0
\]
Saddlepath

Focus

\[ \dot{y} = 0 \quad \dot{x} = 0 \]

\[ y_0 = \text{constant} \quad x_0 = \text{constant} \]
Limit Cycle

\[ \dot{y} = 0 \]

\[ \dot{x} = 0 \]

[Diagram showing a limit cycle with arrows indicating direction of movement along the x and y axes.]
Calculus of Variations

\[ \max_x \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) dt \]

subject to \( x(t_0) = x_0, x(t_1) = x_1 \)

Necessary Conditions: Euler-LaGrange Equation

\[ F_x = F_{xt} + F_{xx} \dot{x} + F_{xx} \ddot{x}, t \in [t_0, t_1] \]
Conversion to Control Framework:

Define: $\dot{x}(t) = u(t)$

$$\max_{x,u} \int_{t_0}^{t_1} F(t, x(t), u(t)) dt$$

subject to: $\dot{x}(t) = u(t), x(t_0) = x_0, x(t_1) = x_1$
Control Theory

• More general than the Calculus of Variations
• Handles Inequality restrictions on Instruments and State Variables
• Similar to Static Methods (LaGrange Method and Kuhn-Tucker Theory)
Objective Functional: \[ \int_{t_0}^{t_1} f(t, x(t), u(t)) \text{d}t \]

Constraint: \[ \dot{x}(t) = g(t, x(t), u(t)) \]

Boundary Conditions: \( t_0, t_1, x(t_0) = x_0 \) fixed

Free Endpoint for \( x \)

Definitions: \( x = \text{State Variable}, u = \text{Instrument} \)

Functional Form Assumption:

\( f() \) and \( g() \) are continuously differentiable

Objective: choose a time path of \( u \) and \( x \), thereby maximizing the functional above.
Necessary Conditions: found by forming the Hamiltonian
\[ H(t,x(t),u(t),\lambda(t)) = f(t,x(t),u(t)) + \lambda g(t,x(t),u(t)) \]

Necessary Conditions:
\[ u : \frac{\partial H}{\partial u} = 0, \quad \dot{\lambda}(t) = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial \lambda} = \dot{x} \]

Boundary (Transversality) Conditions:
\[ x(t_0) = x_0, \quad \lambda(t_1) = 0 \]

Interpreting the Costate Variable: measures the imputed value of stock (state variable) accumulation.
Sufficiency: if \( f() \) and \( g() \) are strictly concave, then the necessary conditions are sufficient, meaning that any path satisfying these conditions does in fact solve the problem posed.

Additional State Variable and Instrument

Objective Functional:
\[
\int_{t_0}^{t_1} f(t, x_1(t), x_2(t), u_1(t), u_2(t))dt
\]

Constraints:
\[
\dot{x}_1(t) = g^1(t, x_1(t), x_2(t), u_1(t), u_2(t)) \\
\dot{x}_2(t) = g^2(t, x_1(t), x_2(t), u_1(t), u_2(t))
\]

Boundary Conditions: \( t_0, t_1, x_1(t_0) = x_{10}, x_2(t_0) = x_{20} \) fixed

Free Endpoints for \( x_1 \) and \( x_2 \)
Optimality Conditions:

Hamiltonian:

\[ H(t,x_1(t),x_2(t),u_1(t),u_2(t),\lambda_1(t), \lambda_2(t)) = \]

\[ f(t,x_1(t),x_2(t),u_1(t),u_2(t)) + \lambda_1 g^1(t,x_1(t),x_2(t),u_1(t),u_2(t)) + \lambda_2 g^2(t,x_1(t),x_2(t),u_1(t),u_2(t)) \]

\[ u_i : \frac{\partial f}{\partial u_i} + \lambda_1 \frac{\partial g^1}{\partial u_i} + \lambda_2 \frac{\partial g^2}{\partial u_i} = 0 \quad i = 1,2 \]

\[ \dot{\lambda}_i = - \frac{\partial H}{\partial x_i} = - \left\{ \frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g^1}{\partial x_i} + \lambda_2 \frac{\partial g^2}{\partial x_i} \right\} \]

\[ \dot{x}_i = \frac{\partial H}{\partial \lambda_i} \]
Boundary (Transversality) Conditions:
\[ x_1(t_0) = x_{10}, \ x_2(t_0) = x_{20}, \ \lambda_1(t_1) = \lambda_2(t_1) = 0 \]

Fixed Endpoint Problem: adds the boundary condition
\[ x(t_1) = x^* \]

More General Problem: adds a terminal-value function and inequality restrictions.

Objective Functional:
\[ \int_{t_0}^{t_1} f(x(t), u(t), t) dt + \varphi(t, x(t_1)) \]

\[ x = n\text{-dimensional vector}, \ u = m\text{-dimensional vector} \]
Constraints:

\[ \dot{x}_i(t) = g_i(t, x, u), i = 1, \ldots, n \]

\[ x_i(t_0) = x_{i0} \text{ fixed}, i = 1, \ldots, n \]

\[ x_i(t_1) = x_{i1} \text{ fixed}, i = 1, \ldots, q \]

\[ x_i(t_1) \text{ free}, i = q + 1, \ldots, r \]

\[ x_i(t_1) \geq 0, i = r + 1, \ldots, s \]

\[ K(x_{s+1}(t), \ldots, x_n(t)) \geq 0 \text{ at } t_1 \]

\[ 1 \leq q \leq r \leq s \leq n \]
Optimality Conditions:

\[ \dot{x}_i = \frac{\partial H}{\partial \lambda_i} = g_i(t, x, u), i = 1, \ldots, n \]

\[ \dot{\lambda}_i = -\left\{ \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \lambda_i \frac{\partial g_j}{\partial x_i} \right\}, j = 1, \ldots, n \]

\[ \frac{\partial f}{\partial u_j} + \sum_{k=1}^{n} \lambda_k \frac{\partial g_k}{\partial u_j} = 0, j = 1, \ldots, m \]

N.B. \( H(t, x^*, u, \lambda) \) is maximized by \( u = u^* \)
Transversality Conditions

(i) $x_i(t_1)$ free: 
$$\lambda_i(t_1) = \frac{\partial \phi}{\partial x_i}$$

(ii) $x_i(t_1) \geq 0$: 
$$\lambda_i(t_1) \geq \frac{\partial \phi}{\partial x_i}, \quad x_i(t_1) \left[ \lambda_i(t_1) - \frac{\partial \phi}{\partial x_i} \right] = 0$$

(iii) $K(x_q(t_1), \ldots, x_n(t_1)) \geq 0$: 
$$\lambda_i(t_1) = \frac{\partial \phi}{\partial x_i} + p \frac{\partial K}{\partial x_i}, \quad i = q, \ldots, n$$
$$p \geq 0, \quad pK = 0$$
(iv) \( K(x_q(t_1),\ldots,x_n(t_1)) = 0: \)

\[
\lambda_i(t_1) = \frac{\partial \phi}{\partial x_i} + p \frac{\partial K}{\partial x_i}, \quad i = q,\ldots,n
\]

(v) \( t_1 \) is free: at \( t_1 \)

\[
f + \sum_{i=1}^{n} \lambda_i g_i + \phi_t = 0
\]

(vi) \( T \geq t_1: \)

\[
f + \sum_{i=1}^{n} \lambda_i g_i + \phi_i \geq 0
\]

at \( t_1 \), with strict equality if \( T > t_1 \), if \( T - t_1 > 0 \) is required
(vii) $K(x_q(t_1),\ldots,x_n(t_1),t_1) \geq 0$:

$$\lambda_i(t_1) = \frac{\partial \varphi}{\partial x_i} + p \frac{\partial K}{\partial x_i}, \quad i = q,\ldots,n$$

$$f + \sum_{i=1}^{n} \lambda_i g_i + \varphi_t + p \frac{\partial K}{\partial t_1} = 0$$

$p \geq 0$, $K \geq 0$, $pK = 0$, $t = t_1$

State Variable Restrictions: $k(t,x) \geq 0$

$$\max \int_{t_0}^{t_1} f(t,x,u)dt + \varphi(x(t_1))$$

subject to

$$\dot{x} = g(t,x,u), x(t_0) = x_0$$

$k(t,x) \geq 0$
Hamiltonian: \( H = f(t, x, u) + \lambda g(x, u, t) + \eta k(t, x) \)

Optimality Conditions:

\[
\begin{align*}
\dot{\lambda} &= -H_x = -(f_x + \lambda g_x + \eta k_x) \\
\lambda(t_1) &= \phi_x(x(t_1)), \eta \geq 0, \eta k = 0
\end{align*}
\]
Discounting, Current Values, and Comparative Dynamics

\[ \int_0^T e^{-rt} f(t, x, u) dt \]

\[ \dot{x} = g(t, x, u), \quad x(0) = x_0 \]

Hamiltonian: \( H = e^{-rt} f(t, x, u) + \lambda g(t, x, u) \)

Optimality Conditions:

\[ u : H_u = e^{-rt} f_u + \lambda g_u \]

\[ \dot{\lambda} = -H_x = -e^{-rt} f_x - \lambda g_x \]

\[ \lambda(T) = 0 \]
Often it is convenient to eliminate the discount factor. The resulting system involves current, rather than, discounted values of various magnitudes.

Hamiltonian: \( H = e^{-rt}[f(t,x,u) + e^{rt}\lambda g(t,x,u)] \)

Define \( m(t) = e^{rt}\lambda(t) \)

Current Value Hamiltonian:

\[
\tilde{H} = e^{rt}H = f(t, x, u) + mg(t, x, u)
\]

Optimality Conditions:

\[
\frac{\partial\tilde{H}}{\partial u} = f_u + mg_u = 0, \quad \dot{m} - rm = -f_x - mg_x
\]
N.B. If \( f() \) and \( g() \) are autonomous, this substitution leads to autonomous transition equations describing the dynamics of the system.

**Equilibria in Infinite Horizon Problems**

\[
\int_0^\infty e^{-rt} f(x,u) dt
\]

\[
\dot{x} = g(x,u), \ x(0) = x_0
\]

Hamiltonian: \( H = f(x,u) + mg(x,u) \)

Optimality Conditions:

\[
H_u = 0, \quad \dot{m} - rm = -H_x
\]
Transversality Conditions:

\[
\lim_{t \to \infty} m(t)x(t) = 0
\]

\[
\lim_{t \to \infty} e^{-rt} m(t)x(t) = 0
\]

N.B. These are NOT necessary conditions.

Problems of this sort, if they are of two dimensions, lead to phase plane analysis.
\dot{x} = 0

\dot{m} = 0
Sufficiency: Arrow and Kurz (1970)

- If the maximized Hamiltonian is strictly concave in the state variables, any path satisfying the conditions above will be sufficient to solve the problem posed.

Hamiltonian: \( H = f(x,u) + mg(x,u) \)

Optimality Conditions: \( H_u = 0, \quad \dot{m} - rm = -H_x \)

\[ \Rightarrow u = \hat{u}(m,x) \]

Maximized Hamiltonian:

\[ H^* = f(x,\hat{u}(m,x)) + mg(x,\hat{u}(m,x)) \]
Steady-State Equilibrium

Eliminate Instruments Using Necessary Conditions:

\[ v = \tilde{v}(x, z) \]
\[ f(x^*, z) = 0, g(x^*, z) = 0, \dot{x} = 0 \]

Solve Using Cramér’s Rule:

\[ \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial z_1} dz_1 = 0 \]
\[ \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 + \frac{\partial g}{\partial z_1} dz_1 = 0 \]
\[
\left[ \begin{array}{cc}
\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\
\frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2}
\end{array} \right] \left[ \begin{array}{c}
dx_1 \\
\frac{dz_1}{dz_1}
\end{array} \right] = - \left[ \begin{array}{c}
\frac{\partial f}{\partial z_1} \\
\frac{\partial g}{\partial z_1}
\end{array} \right]
\]

\[
\begin{align*}
\frac{\partial x_1}{\partial z_1} &= \begin{vmatrix}
-\frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial x_2} \\
-\frac{\partial g}{\partial z_1} & \frac{\partial g}{\partial x_2}
\end{vmatrix}, \\
\frac{\partial x_2}{\partial z_1} &= \begin{vmatrix}
-\frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial x_2} \\
-\frac{\partial g}{\partial z_1} & \frac{\partial g}{\partial x_2}
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \frac{\partial f}{\partial x_2} \\
\frac{\partial g}{\partial x_1} &= \frac{\partial g}{\partial x_2}
\end{align*}
\]
Example II: Cash Flow Maximization by the Firm

$$\max \int_{0}^{\infty} [P[F(L(t)) - T(N(t))] - WL(t)]e^{-rt}dt$$

subject to \[\dot{L}(t) = N(t) - QL, 0 < Q < 1\]

Definitions: \[L = \text{Labor}, P = \text{Output Price}, N = \text{Hires},\]
\[r = \text{Discount Rate}, Q = \text{Quit Rate},\]
Present Value Hamiltonian:

\[ H = P[F(L(t)) - T(N(t))] - WL(t)] + \lambda(N - QL) \]

Optimality Conditions:

\[ PT'(N) = \lambda \]

\[ -\frac{\partial H}{\partial L} = \dot{\lambda} \Rightarrow \dot{\lambda} - r\lambda = W - PF'(L) + Q\lambda \]

\[ \frac{\partial H}{\partial \lambda} = \dot{L} \Rightarrow \dot{L} = N - QL \]

\[ \lim_{t \to \infty} \lambda e^{-rt} L = 0 \]
Inequality Constraints

Suppose that $N \geq 0$

The firm is now unable to fire workers. Now the problem must be rewritten to explicitly incorporate the new restriction.
Hamiltonian:

\[ H = P[F(L(t)) - T(N(t))] - WL(t) + \lambda(N - QL) \]

LaGrangian:

\[ L = P[F(L(t)) - T(N(t))] - WL(t) + \lambda(N - QL) + \pi N \]

Optimality Conditions:

\[ PT'(N) + \pi = \lambda \]
\[ \pi \geq 0, \pi N = 0 \]
\[ -\frac{\partial H}{\partial L} = \dot{\lambda} \Rightarrow \dot{\lambda} = W - PF'(L) + Q\lambda \]
\[ \frac{\partial H}{\partial \lambda} = \dot{L} \Rightarrow \dot{L} = N - QL \]
\[ \lim_{t \to \infty} \lambda e^{-rt} L = 0 \]
Discrete Time Optimal Control

State Dynamics: \( x_{t+1} - x_t = f(x_t, u_t, t), \) \( x_0 \) given

Objective Functional: \( J = S(x_T) + \sum_{t=0}^{T-1} u(x_t, u_t t) \)

Hamiltonian: \( H(x_t, u_t, p_{t+1}, t) = u(x_t, u_t, t) + p_{t+1}' f(x_t, u_t, t) \)

Optimality Conditions:

\[
H_u = 0
\]

\[
x_{t+1} - x_t = \frac{\partial H}{\partial p_{t+1}} = f(x_t, u_t, t)
\]

\[
p_{t+1} - p_t = -\frac{\partial H}{\partial x_t} = -\frac{\partial u}{\partial x_t} - p_{t+1}' \frac{\partial f}{\partial x_t}
\]
Boundary Conditions:

\[ x_0 \text{ given} \quad p_T = \frac{\partial S}{\partial x_T} \]

**Discrete Calculus of Variations**

\[
\max \sum_{t=0}^{\infty} \beta^t g(x_{t+1}, x_t)
\]

The trick here is to differentiate the objective function with respect to \( x_t \) at time \( t \) and \( t-1 \).
Objective Function at time t: $\beta^t g(x_{t+1}, x_t)$

Optimality Condition at time t: $\beta^t g_{x_t} (x_{t+1}, x_t)$

Optimality Condition at t-1: $\beta^{t-1} g_{x_t} (x_t, x_{t-1})$

Complete Condition:

$$\beta^t g_{x_t} (x_{t+1}, x_t) + \beta^{t-1} g_{x_t} (x_t, x_{t-1}) = 0$$

$$g_{x_t} (x_{t+1}, x_t) + \beta^{t-1} g_{x_t} (x_t, x_{t-1}) = 0$$

Work out the case where the objective function is $g(x_{t+1}, x_t, x_{t-1})$. 
Transversality Condition:

\[ \lim_\limits_{T \to \infty} \beta^T \frac{\partial g}{\partial x_T} x_T = 0 \]
Example III: Cash Flow Maximization by a Firm

\[
\max \sum_{t=0}^{\infty} \beta^t \left[ P[f(L_t) - C(L_{t+1} - L_t)] - WL_t \right], 0 < \beta < 1
\]

Euler Equation:

\[
\beta^t [P[f'(L_t) + C'(L_{t+1} - L_t)] - W] - \beta^{t-1} PC'(L_t - L_{t-1}) = 0
\]

\[
P[f'(L_t) + C'(L_{t+1} - L_t)] - W - \beta^{-1} PC'(L_t - L_{t-1}) = 0
\]
Dynamic Programming

- Alternative way to solve intertemporal problems
- Equivalent in many contexts to methods already seen
- Drawn from Burnside “Dynamic Optimization”
\[
\max \sum_{t=0}^{T} \beta^t u(c_t) + \tilde{V}_0(k_{T+1})
\]

subject to \[c_t + k_{t+1} - (1 - \delta)k_t = f(k_t)\]

Lagrangean Formulation:

\[
\sum_{t=0}^{T} \beta^t u(c_t) + \tilde{V}_0(k_{T+1}) + \sum_{t=0}^{T} \tilde{\lambda}_t [f(k_t) + (1 - \delta)k_t - c_t - k_{t+1}]
\]
Optimality Conditions:

\[ c_t : \beta^t u'(c_t) - \tilde{\lambda}_t = 0 \]
\[ k_{t+1} : \tilde{\lambda}_{t+1} [f'(k_{t+1}) + (1 - \delta)] - \tilde{\lambda}_t = 0, 0 < t < T \]
\[ k_{T+1} : \tilde{V}'_0(k_{T+1}) - \tilde{\lambda}_t \leq 0, \left[ \tilde{V}'_0(k_{T+1}) - \tilde{\lambda}_t \right] k_{T+1} = 0 \]
\[ c_t + k_{t+1} - (1 - \delta)k_t = f(k_t) \]

Eliminate the multiplier to get
\[
\beta u'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)] = u'(c_t)
\]
\[
\tilde{V}_0'(k_{T+1}) - \tilde{\lambda}_t \leq 0, \left[ \tilde{V}_0'(k_{T+1}) - \tilde{\lambda}_t \right] k_{T+1} = 0
\]
\[
c_t + k_{t+1} - (1 - \delta)k_t = f(k_t)
\]

Problem can be solved recursively:

• First solve the problem at \( t = T \)

• Choose \( c_T \) and \( k_{T+1} \) to maximize
\[ \beta^T u(c_T) + \tilde{V}_0(k_{T+1}) \]

subject to \[ c_T + k_{T+1} - (1 - \delta)k_T = f(k_T) \quad k_T \text{ given} \]

\[ \Rightarrow c_T = c_T(k_T), k_{T+1} = k_{T+1}(k_T) \]

Now solve the period T-1 problem

• Choose \( c_{T-1} \) and \( k_T \) to maximize

\[ \beta^{T-1} u(c_{T-1}) + \beta^T u(c_T(k_T)) + \tilde{V}_0(k_{T+1}(k_T)) \]

subject to \[ c_{T-1} + k_T - (1 - \delta)k_{T-1} = f(k_{T-1}) \quad k_{T-1} \text{ given} \]
Continue solving backwards to time 0.

The same optimality conditions arise from the problem

\[
V_1(k_T) = \max_{c_T, k_{T+1}} u(c_T) + \beta V_0(k_{T+1})
\]

subject to \( c_T + k_{T+1} - (1-\delta)k_T = f(k_T) \)

\[
V_0(k_{T+1}) = \tilde{V}_0(k_{T+1})/\beta^{T+1}
\]
Optimality Conditions:

\[ u'(c_T) = \lambda_T \]
\[ \beta V'_0(k_{T+1}) - \lambda_T \leq 0, [\beta V'_0(k_{T+1}) - \lambda_T] k_{T+1} = 0 \]
\[ c_T + k_{T+1} - (1 - \delta)k_T = f(k_T) \]

These are the same conditions as before if we define

\[ \lambda_T = \tilde{\lambda}_T / \beta^T \]

Envelope Theorem implies

\[ V'_1(k_T) = \lambda_T [f'(k_T) + (1 - \delta)] \]
Given the constraint and $k_T$, $V_1(k_T)$ is the maximized value of

$$u(c_T) + \beta V_0(k_{T+1})$$

Period $T-1$ problem is equivalent to maximizing

$$u(c_{T-1}) + \beta V_1(k_T)$$

with the same constraint at $T-1$ and $k_{T-1}$ given.

$$V_2(k_{T-1}) = \max_{c_{T-1}, k_T} u(c_{T-1}) + \beta V_1(k_T)$$

subject to

$$c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$$
Optimality Conditions:

\[ u'(c_{T-1}) = \lambda_{T-1} \]
\[ \beta V_1'(k_T) = \lambda_{T-1} \]
\[ c_{T-1} + k_T - (1 - \delta)k_{T-1} = f(k_{T-1}) \]

The envelope theorem can be used to eliminate \( V_1' \)

\[ u'(c_{T-1}) = \lambda_{T-1} \]
\[ \beta \lambda_T [f'(k_T) + (1 - \delta)] = \lambda_{T-1} \]
\[ c_{T-1} + k_T - (1 - \delta)k_{T-1} = f(k_{T-1}) \]
The period T-1 envelope condition is

\[ V_2'(k_{T-1}) = \lambda_{T-1} [f'(k_{T-1}) + (1 - \delta)] \]

This process can be continued giving the following Bellman Equation.

\[ V_{j+1}(k_{T-j}) = \max_{c_T, k_T} u(c_{T-j}) + \beta V_j(k_{T-j+1}) \]

subject to \[ c_{T-j} + k_{T-j+1} - (1 - \delta)k_{T-j} = f(k_{T-j}) \] \( k_{T-j} \) given

Bellman’s Principle of Optimality

• The fact that the original problem can be written in this recursive way