Time Series Analysis

First set of assignments

1. The stochastic process \( \{ \varepsilon_t \} (t = 1, 2, \cdots) \) consists of independent random variables \( \varepsilon_t \sim N(0, 1) \). Compute the probability \( P(\varepsilon_t \leq 0 \cap \varepsilon_{t+1} > 1.96 \cap \varepsilon_{t+2} \leq -1.96) \).

2. Write the joint density \( f_{\varepsilon_t \varepsilon_{t+1}}(\varepsilon_t, \varepsilon_{t+1}) \). Interpret your result.

3. Write the conditional density \( f_{\varepsilon_{t+1}|\varepsilon_t}(\varepsilon_{t+1}|\varepsilon_t) \).

4. Denote a realisation of the stochastic process \( \{ \varepsilon_t \} \) as \( \{ x_1, x_2, \cdots, x_T \} \).
   Write down the joint density function of the random vector \( \varepsilon = \{ \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_T \} \) evaluated at \( \{ x_1, x_2, \cdots, x_T \} \).

   Since the random vector \( \varepsilon = \{ \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_T \} \) is jointly normally distributed you can use the multivariate normal density which is generally written as

   \[
   f_{\varepsilon} = 2\pi^{-n/2} |\Omega|^{-0.5} \exp\left[ \frac{(-\mu)'\Omega^{-1}(x-\mu)}{2} \right]
   \]

   What is in our example \( n, \varepsilon, \mu \) and \( \Omega \)?

5. Is the process \( \{ \varepsilon_t \} \) weakly stationary?

6. Is the process \( \{ \varepsilon_t \} \) strictly stationary?

7. A new stochastic process \( \{ Y_t \} \) is generated as \( Y_t = a + b \cdot \varepsilon_t \)
   The joint distribution of \( Y = (Y_1, Y_2, \cdots, Y_T) \) is still the multivariate normal (see 4.)
   What is \( \mu \) and \( \Omega \) now?

8. \( \{ X_t \} \) denotes a stochastic process. We have \( E(X_t) = E(X_{t+1}) = 2 \)
   \( \text{cov}(X_t, X_{t+1}) = 2 \) and \( \text{var}(X_t) = \text{var}(X_{t+1}) = 1 \)

   using \( A = \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{bmatrix} \) we generate two new random variables \( Z_1, Z_2 \) by

   \[
   Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = A \cdot \begin{bmatrix} X_t \\ X_{t+1} \end{bmatrix}
   \]

   compute \( E(Z) \) and \( \text{cov}(Z) = \begin{bmatrix} \text{var}(Z_1) & \text{cov}(Z_1, Z_2) \\ \text{cov}(Z_1, Z_2) & \text{var}(Z_2) \end{bmatrix} \)
Solutions to the first set of assignments:

1. \( P(\varepsilon_t \leq 0) \cdot P(\varepsilon_{t+1} > 1.96) \cdot P(\varepsilon_t \leq -1.96) = 0.5 \cdot 0.025 \cdot 0.025 = 0.0003125 \)

8. \( E(Z) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \)

\( \text{cov}(Z) = \begin{bmatrix} 1.42 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \)